# THE TRANSFORMATION $C$ OF NETS IN HYPERSPACE*

BY

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## 1. Introduction

It is the purpose of this paper to extend some of the ideas related to the transformation† $C$ of nets in projective space of three dimensions to nets in hyperspace. In three dimensions two nets $N_x$ and $N_y$ are said to be in relation $C$ if the developables of the congruence $G$ of lines joining corresponding points $x$ and $y$ intersect the two sustaining surfaces in those nets, provided however that neither surface is a focal surface of the congruence. If $N_x$ and $N_y$ are in relation $C$, the tangents at $x$ and $y$ to corresponding curves of the nets intersect.

We shall say that two nets $N_x$ and $N_y$ in space $S_n$ of $n$ dimensions are *in relation* $C$ if the two sustaining surfaces $S_x$ and $S_y$ are such that corresponding tangent planes intersect in a line, and if the developable surfaces of the congruence $G$ of lines $g$ joining corresponding points $x$ and $y$ intersect the two surfaces in the nets. It is to be understood that the two sustaining surfaces are not the focal surfaces of $G$. Not all nets $N_x$ in $S_n$ for $n \geqq 4$ can have a $C$ transform $N_y$. A net in $S_n$ which permits of having a $C$ transform will be called *a $C$ net.*

We derive necessary and sufficient conditions that a non-conjugate net be a $C$ net. Another geometrical interpretation is given for two covariant points found by Bompiani.‡

Let $S_x$ and $S_y$ be two surfaces in the same space $S_n$ of $n$ dimensions and with their points in one-to-one point correspondence. Let the parametric equations of these surfaces be

$$x^{(i)} = x^{(i)}(u, v), \quad y^{(i)} = y^{(i)}(u, v) \qquad (i = 1, 2, 3, \cdots, n+1),$$

the parameters being so chosen that corresponding points have the same curvilinear coördinates. Suppose furthermore that the tangent planes to $S_x$ and $S_y$ at corresponding points intersect in a line. There exist therefore scalar functions $m, s, f, A$, etc., such that

* Presented to the Society, June 13, 1931; received by the editors March 1, 1931.

† V. G. Grove, *Transformations of nets*, these Transactions, vol. 30 (1928), p. 483.

‡ E. Bompiani, *Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali lineari ed omogenee*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), pp. 610–636. In particular see p. 634. Hereafter referred to as Bompiani, *Surfaces*.

$$
(1) \qquad \begin{aligned} y_u &= mx_u + sx_v + fx + Ay, \\ y_v &= tx_u + nx_v + gx + By. \end{aligned}
$$

Any point $z$ on the line $g$ joining corresponding points $x$ and $y$ is defined by an expression of the form

$$
z = y + \lambda x.
$$

Consider the surface $S_z$ generated by the point $z$, and a curve $C$ on $S_z$ with parametric equations

$$
u = u(t), \quad v = v(t).
$$

The tangent line to $C$ at $z$ is determined by $z$ and $z' = dz/dt$ where

$$
(2) \qquad z' = [(m + \lambda)u' + tv']x_u + [su' + (n + \lambda)v']x_v + (\ )x + (\ )y.
$$

Hence the line $g$ generates a congruence in the ordinary sense, that is, the lines of the two-parameter families of lines may be grouped into two one-parameter families of developable surfaces. The curves on $S_x$ corresponding to these developables are defined by

$$
(3) \qquad sdu^2 - (m - n)dudv - tdv^2 = 0.
$$

We shall assume that these curves are not indeterminate and are distinct. By a change of parameters we may make the curves determined by (3) parametric. Let us suppose that this transformation has been made. *If two surfaces are such that their parametric nets are in relation $C$, the functions $x$ and $y$ determining the surfaces satisfy equations of the form*

$$
(4) \qquad \begin{aligned} y_u &= mx_u + fx + Ay, \\ y_v &= nx_v + gx + By, \quad mn(m - n) \neq 0. \end{aligned}
$$

From (2) we find that the focal points of $g$ are defined by

$$
(5) \qquad \tau = mx - y, \quad \tau' = nx - y.
$$

The nets defined by $\tau$ and $\tau'$ will be called *the focal nets* of $G$. The tangent to the curve $v = \text{const.}$ ($u = \text{const.}$) on the focal surface $S_{\tau'}$ ($S_\tau$) will be called *the minus first (first) derived line of* $g$, and the congruences generated by them *the minus first (first) derived congruences of* $G$.

If we differentiate equations (4) with respect to $v$ and $u$ respectively, we find that $x$ and $y$ satisfy an equation of the form

$$
(6) \qquad x_{uv} = ax_u + bx_v + cx + My
$$

wherein $a, b, c, M$ are defined by the formulas

$$(7)\qquad \begin{aligned}(m-n)a &= g+Bm-m_v, \quad (m-n)c = Bf-Ag+g_u-f_v,\\ (n-m)b &= f+An-n_u, \quad (m-n)M = B_u-A_v.\end{aligned}$$

Since $y$ is not in the tangent plane to $S_x$ at $x$, *the vanishing of $M$ is a necessary and sufficient condition that the net $N_x$ be conjugate.*

If the coefficients of the equations corresponding to (4) and (6) with the rôles of $x$ and $y$ interchanged are denoted by $\bar{m}$, $\bar{f}$, etc., we find that

$$\bar{m} = 1/m,\ \bar{n} = 1/n,\ \bar{f} = -A/m,\ \bar{g} = -B/n,\ \bar{A} = -f/m,\ \bar{B} = -g/n,$$

$$\bar{a} = a+m_v/m,\ \bar{b} = b+n_u/n,$$

$$\bar{c} = -m[aA/m+bB/n+fB/(mn)-M-(A/m)_v],$$

$$(8)\qquad \bar{M} = -m[af/m+bg/n+fg/(mn)-c-(f/m)_v].$$

## 2. The relation $R$ in $S_n$

Denote by $R^{(v)}$ the ruled surface formed by the tangents to the curves $v=\text{const.}$ at the points where they meet a fixed curve $u=\text{const.}$ A ruled surface $R^{(u)}$ may be defined similarly.

Let $l'$ be any line lying in the tangent plane to $S_x$ at $x$, but not passing through $x$. Let $l$ be a line passing through $x$ but not lying in the tangent plane at $x$. The line $l'$ intersects the tangent to the curves $v=\text{const.}$ and $u=\text{const.}$ in points $r$ and $s$ respectively. If the tangent planes to $R^{(v)}$ and $R^{(u)}$ at $r$ and $s$ respectively intersect in the line $l$ the given lines $l$ and $l'$ will be said to be *in relation* $R$ with respect to $N_x$.*

The points $r$ and $s$ are defined by expressions of the form

$$(9)\qquad r = x_u-\lambda x,\quad s = x_v-\mu x.$$

A point in the tangent plane to $R^{(v)}$ at $r$ is

$$(10)\qquad r_v+\alpha r+\beta x = x_{uv}+\alpha x_u-\lambda x_v+(\beta-\alpha\lambda-\lambda_v)x.$$

A point in the tangent plane to $R^{(u)}$ at $s$ is

$$(11)\qquad S_u+\alpha' s+\beta' x = x_{uv}+\alpha' x_v-\mu x_u+(\beta'-\alpha'\mu-\mu_u)x.$$

From (10) and (11) we observe that the tangent planes to $R^{(v)}$ and $R^{(u)}$ at $r$ and $s$ intersect in a line joining $x$ to $z$ defined by

$$(12)\qquad z = x_{uv}-\mu x_u-\lambda x_v.$$

* In a footnote on p. 86 of his paper *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), Green defined the relation $R$ between two lines with respect to a net in $S_3$. The definition we have used for the relation $R$ between two lines with respect to a net in $S_n$ reduces to Green's definition when $n=3$. It is to be noted however that not all lines $l$ in $S_n$ for $n>3$ and protruding from the surface at $x$ have a line $l'$ in relation $R$ to $N_x$.

*The line $l$ in relation $R$ to $l'$ therefore joins $x$ to the point $z$ defined by* (12).

If $M \neq 0$, we see readily from (6) that the line $g'$ in relation $R$ to $g$ with respect to $N_x$ joins the points

$$\rho = x_u - bx, \quad \sigma = x_v - ax. \tag{13}$$

The line $\bar{g}'$ in relation $R$ to $g$ with respect to $N_y$ is determined by the points $\bar{\rho}$, $\bar{\sigma}$ defined by

$$\bar{\rho} = y_u - \bar{b}y, \quad \bar{\sigma} = y_v - \bar{a}y. \tag{14}$$

An examination of equations (5) readily shows that *the derived lines of $g$ intersect the tangents to the curves of $N_x$ and $N_y$ in the points determining the lines $g'$ and $\bar{g}'$ in relation $R$ to $N_x$ and $N_y$.* If $N_x$ and $N_y$ are in relation* $F$, that is, if they are in relation $C$ and are both conjugate nets, the derived lines intersect the tangents to the curves of the nets in the focal points of these tangents.

The tangent to the curve $u = \text{const.}$ at $\rho$ on the surface generated by that point intersects $g$ in the point

$$(ab + c - b_v)x + My. \tag{15}$$

Similarly the tangent to $v = \text{const.}$ at $\sigma$ intersects $g$ in the point

$$(ab + c - a_u)x + My. \tag{16}$$

The points defined by (15) and (16) coincide if $N_x$ is conjugate, or if

$$a_u - b_v = 0. \tag{17}$$

### 3. The third-order differential equations of the problem

Let us assume that the nets $N_x$ and $N_y$ are not conjugate. If we differentiate equation (6) with respect to $u$ and $v$, and use (4) and (6), we find that the functions $x$ must satisfy two differential equations of the form

$$\begin{aligned} x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx, \\ x_{uvv} &= E'x_{uv} + bx_{vv} + G'x_u + H'x_v + J'x, \end{aligned} \tag{18}$$

wherein

$$\begin{aligned} E &= b + A + M_u/M, & E' &= a + B + M_v/M, \\ G &= a_u + c + mM - a(E - b), & H' &= b_v + c + nM - b(E' - a), \\ H &= b_u - b(E - b), & G' &= a_v - a(E' - a), \\ J &= c_u + fM - c(E - b), & J' &= c_v + gM - c(E' - a). \end{aligned} \tag{19}$$

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* L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34.

Similar third-order differential equations are satisfied by the functions $y$. Hence *if two surfaces $S_x$ and $S_y$ are in one-to-one point correspondence with corresponding tangent planes intersecting in a line, and if the nets on $S_x$ and $S_y$ which are in relation C are parametric, the functions $x$ (and $y$) satisfy two third-order differential equations of the type* (18). Differential equations of this type have been studied by Bompiani* and Lane.†

Conversely suppose the coördinates $x$ defining a surface $S_x$ satisfy a pair of differential equations of the form (18). In case the two-osculating space $S(2, 0)$ of $S_x$ at $x$ determined by the points $x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x$ is an $S_5$, the integrability conditions‡ of system (18) are

$$\begin{aligned} aE' + G' - a^2 - a_v = 0,\ \ bE + H - b^2 - b_u &= 0, \\ bE' + E_u' + H' &= aE + E_v + G, \\ E'G + bG' + G_u' + J' &= EG' + aG + G_v, \\ EH' + aH + H_v + J &= E'H + bH' + H_u', \\ E'J + bJ' + J_u' &= EJ' + aJ + J_v. \end{aligned} \tag{20}$$

We shall show that *every net $N_x$ whose defining functions $x$ satisfy differential equations of the form* (18) *is a C net*. Since $x$ and $y$ must satisfy equations of the form (4) it follows that the point $y$ is defined by an expression of the form

$$y = x_{uv} - \lambda x_u - \mu x_v + hx. \tag{21}$$

Case I. Suppose that $S(2, 0)$ is an $S_5$. Differentiating (21) with respect to $u$ and $v$ we find readily that

$$\begin{aligned} y_u = {} & (a - \lambda)x_{uu} + [G + h - \lambda_u + \lambda(E - \mu)]x_u + [H + \mu(E - \mu) - \mu_u]x_v \\ & + [J - h(E - \mu) + h_u]x + (E - \mu)y, \\ y_v = {} & (b - \mu)x_{vv} + [G' + \lambda(E' - \lambda) - \lambda_v]x_u + [H' + h - \mu_v + \mu(E' - \lambda)]x_v \\ & + [J' - h(E' - \lambda) + h_v]x + (E' - \lambda)y. \end{aligned} \tag{22}$$

Hence $N_y$ will be in relation $C$ to $N_x$ if and only if $\lambda = a$, $\mu = b$, $h$ arbitrary. The functions $x$ and $y$ satisfy equations of the form (4) with

$$\begin{aligned} m &= aA + G - a_u + h, & n &= bB + H' - b_v + h, \\ f &= J - Ah + h_u, & g &= J' - Bh + h_v, \\ A &= E - b, & B &= E' - a. \end{aligned} \tag{23}$$

* Bompiani, *Surfaces*, p. 632.

† E. P. Lane, *Integral surfaces of pairs of partial differential equations of the third order*, these Transactions, vol. 32 (1930), pp. 782–793.

‡ Bompiani, *Surfaces*, p. 632.

*There exists*, therefore, *in this case one and only one line passing through $x$ and not lying in the tangent plane to $S_x$ at $x$, such that every point $y$ on the line, not a focal point of the line, generates a net $N_y$ in relation $C$ to $N_x$. The space $S_n$ is such that $n \geqq 5$.*

Case II. Suppose that $S(2, 0)$ is an $S_4$. It follows that the functions $x$ satisfy an equation of the form

$$P'x_{uu} + Q'x_{uv} + R'x_{vv} + L'x_u + N'x_v + K'x = 0$$

such that not both $P'$ and $R'$ are zero. To fix the notation, suppose that $R' \neq 0$. The functions $x$ therefore satisfy a system of differential equations of the form

$$\begin{aligned} x_{vv} &= Px_{uu} + Qx_{uv} + Lx_u + Nx_v + Kx, \\ x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx, \\ x_{uvv} &= D'x_{uu} + E'x_{uv} + G'x_u + H'x_v + J'x. \end{aligned} \tag{24}$$

Subcase (a). Suppose that $x$ does not satisfy a differential equation of the form

$$x_{uuu} = \alpha x_{uu} + \beta x_{vv} + \delta x_u + \epsilon x_v + \phi x. \tag{25}$$

Some of the integrability conditions of system (24) are

$$P = D' = 0, \quad aE' + G' - a^2 - a_v = 0, \quad aQ + L = 0.$$

From (21) we find that

$$\begin{aligned} y_u = {} & (a - \lambda)x_{uu} + [G + h - \lambda_u + \lambda(E - \mu)]x_u + [H + \mu(E - \mu) - \mu_u]x_v \\ & + [J - h(E - \mu) + h_u]x + (E - \mu)y, \\ y_v = {} & [G' - \lambda_v - \mu L + \lambda(E' - \lambda - \mu Q)]x_u + [H' - \mu N - \mu_v + h \\ & + \mu(E' - \lambda - \mu Q)]x_v + [J' - \mu K + h_v - h(E' - \lambda - \mu Q)]x \\ & + (E' - \lambda - \mu Q)y. \end{aligned} \tag{26}$$

Hence $N_y$ will be in relation $C$ to $N_x$ if and only if

$$\lambda = a, \quad \mu_u + \mu^2 - E\mu - H = 0, \quad h \text{ arbitrary}.$$

The functions $x$ and $y$ satisfy equations of the form (4) with

$$\begin{aligned} m &= aA + G - a_u + h, & n &= H' - \mu B - \mu N - \mu_v + h, \\ f &= J - Ah + h_u, & g &= J' - \mu K + h_v - Bh, \\ A &= E - \mu, & B &= E' - a - \mu Q. \end{aligned} \tag{27}$$

*Hence, in this case, there exist lines $g$ through $x$ such that any point $y$ on any one of the lines generates a net $N_y$ in relation $C$ to $N_x$. These lines $g$ belong to a pencil* *of lines with center at $x$ and in the plane determined by the points $x$, $x_v$, $x_{uv}-ax_u$. The lines $g$ at a point $x$ are projectively related to the lines $g$ through any other point of the curve $v$ = const. through $x$. The space $S_n$ is such that $n \geqq 5$.*

Subcase (b). Suppose that $x$ satisfies a system of differential equations of the form

$$
\begin{aligned}
x_{vv} &= Px_{uu} + Qx_{uv} + Lx_u + Nx_v + Kx,\\
x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx,\\
x_{uvv} &= D'x_{uu} + E'x_{uv} + G'x_u + H'x_v + J'x,\\
x_{uuu} &= \alpha x_{uu} + \beta x_{uv} + \delta x_u + \epsilon x_v + \phi x.
\end{aligned} \tag{28}
$$

Some of the integrability conditions of system (28) are

$$
\begin{gathered}
a^2 + a_v + ED' + HP = \alpha D' + D_u' + aE' + G,\\
D' = L + aQ + \alpha P + P_u.
\end{gathered} \tag{29}
$$

We may show from (21) that the net $N_y$ is in relation $C$ to $N_x$ if and only if

$$
\begin{gathered}
\lambda = a,\ D' - \mu P = 0,\ \mu_u + \mu^2 - E\mu - H = 0,\\
G' - a_v - a^2 + aE' - \mu(L + aQ) = 0.
\end{gathered} \tag{30}
$$

The value of $\mu$ determined by the second of (30), if substituted in the last two of (30), gives two conditions on the coefficients of system (28). These conditions are however a result of the integrability conditions (29). *Hence*, in this case, *there is a unique line joining $x$ to*

$$y = x_{uv} - ax_u - D'x_v/P + hx$$

*any point of which generates a net in relation $C$ to $N_x$. The space $S_n$ is an $S_4$.*

The functions $x$ and $y$ satisfy equations of the form (4) with

$$
\begin{aligned}
m &= aA + G_u - a_u + h, & n &= H' - D'N/P - (D'/P)_v + D'B/P + h,\\
f &= J - Ah + h_u, & g &= J' - D'K/P - Bh - h_v,\\
A &= E - D'/P, & B &= E' - a - D'Q/P.
\end{aligned}
$$

We may state the results of this section as follows: *A non-conjugate net is a $C$ net if and only if the functions $x$ defining the net satisfy a pair of equations of the form* (18).

## 4. The Bompiani transforms of a $C$ net

Suppose the functions $x$ determining a net $N_x$ satisfy differential equations of the form (18). Bompiani* has shown that there exist two covariant points,

* Bompiani, *Surfaces*, pp. 634–635.

one on each tangent line to the curves of the net, characterized in the following way: The point

$$\rho = x_u - bx$$

is the only point on the tangent to $v = \text{const.}$ on $S_x$ generating a surface for which the osculating plane to the curve $u = \text{const.}$ at $\rho$ lies in the $S_3$ determined by the points $x$, $x_u$, $x_v$, $x_{uv}$ and tangent to the ruled surface $R^{(v)}$ along the generator through $x$. The point

$$\sigma = x_v - ax$$

has a similar characterization. We shall call the point $\rho(\sigma)$ the *minus first (first) Bompiani transform of* $x$, and the nets described by $\rho(\sigma)$ the *minus first (first) Bompiani transform of* $N_x$.

Suppose that the functions $x$ satisfy a system of differential equations of the form (24), but no equation of the form (25). We find that the point $\rho$, defined by

$$\rho = x_u - \mu x,$$

generates a surface of the type described above for every value of $\mu$. The point $\sigma$ defined by

$$\sigma = x_v - \dot{a}x$$

is the only point on the tangent to the curve $u = \text{const.}$ at $x$ generating a surface of the desired type. We shall say in this case that *the minus first Bompiani transform of* $x$ $(N_x)$ *is indeterminate.* The first Bompiani transform of $x$ $(N_x)$ is the point $\sigma$ (net $N_\sigma$). Hence *a necessary and sufficient condition that a net permit of having Bompiani transforms is that the given net be a C net.*

We may state some of the results of this and the preceding section in the following theorem:

*Let there be given a net* $N_x$ *in* $S_n$ *whose sustaining surface* $S_x$ *is such that its two-osculating space* $S(2, 0)$ *is an* $S_4$ *or an* $S_5$. *There exists a unique line g through x lying in the* $S(2, 0)$ *of* $S_x$ *at x such that every point y on g generates a net* $N_y$ *in relation C to* $N_x$ *if and only if the given net admits of having uniquely determined Bompiani transforms. If one of the Bompiani transforms is indeterminate there exists a pencil of lines g through x with the above property. The line g is the line in relation R with respect to* $N_x$ *to the line joining the Bompiani transforms of x.* Moreover *the minus first (first) derived line of g intersects the tangent plane of the sustaining surface in the minus first (first) Bompiani transform of x.*

The Bompiani transforms of a $C$ net are $C$ nets. The functions $\rho$ and $\sigma$ satisfy differential equations of the form (18). As may readily be verified the first Bompiani transform of $\rho$ is the point $\bar{\sigma}$ defined by

$$\bar{\sigma} = \rho_v - a\rho,$$

and the minus first Bompiani transform of $\sigma$ is the point $\bar{\rho}$ defined by

$$\bar{\rho} = \sigma_u - b\sigma.$$

*The points $\bar{\rho}$ and $\bar{\sigma}$ lie on the line $g$, and coincide with the points* (15) *and* (16) *respectively.* From (17) we see that these points coincide* if and only if

$$a_u - b_v = 0. \tag{17bis}$$

* Bompiani, *Surfaces*, p. 635.

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